Class of invariants for the 2D time-dependent Landau problem and harmonic oscillator in a magnetic field

Gaetano Fiore
Dip. di Matematica e Applicazioni,
Università Federico II, v. Claudio 21, 80125 Napoli,
and I.N.F.N., Sezione di Napoli, Italy

Laure Gouba
The Abdus Salam International Centre for Theoretical Physics (ICTP),
Strada Costiera 11, I-34151 Trieste Italy
Email: lgouba@ictp.it

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Abstract

We consider an isotropic two dimensional harmonic oscillator with arbitrarily time-dependent mass $M(t)$ and frequency $\Omega(t)$ in an arbitrarily time-dependent magnetic field $B(t)$. We determine two commuting invariant observables (in the sense of Lewis and Riesenfeld) L, I in terms of some solution of an auxiliary ordinary differential equation and an orthonormal basis of the Hilbert space consisting of joint eigenvectors φ_λ of L, I. We then determine time-dependent phases $\alpha_\lambda(t)$ such that the $\psi_\lambda(t) = e^{i\alpha_\lambda}\varphi_\lambda$ are solutions of the time-dependent Schrödinger equation and make up an orthonormal basis of the Hilbert space. These results apply, in particular to a two dimensional Landau problem with time-dependent M, B, which is obtained from the above just by setting $\Omega(t) \equiv 0$. By a mere redefinition of the parameters, these results can be applied also to the analogous models on the canonical non-commutative plane.

1 Introduction

The study of time-dependent quantum problems has been of great interest in the literature since the work of Lewis and Riesenfeld [1, 2], in the late 1960s. They developed a completely general approach to the evolution generated by a time-dependent Hamiltonian $H(t)$; one re-obtains the results of the adiabatic approximation in the case of slowly varying $H(t)$, and the results of the "sudden" approximation in the case of fast or even discontinuously varying $H(t)$. In Refs.[1] and [2], the method was applied to a one-dimensional harmonic oscillator with a t-dependent frequency and to a charged particle in a t-dependent, axially symmetric electromagnetic field in three dimensions. For zero electric field, the latter reduces to a Landau problem with time-dependent magnetic field. Several other applications of the method have been worked out later; we just mention the application in [3] to the anisotropic three-dimensional harmonic oscillator, the ones [4] to the damped harmonic oscillator and to a [5] linear (in space) potential in one dimension. In the present work, we apply the approach of Lewis and Riesenfeld to a two-dimensional isotropic harmonic oscillator with time-dependent mass M and frequency Ω in a time-dependent (but space-independent magnetic field) B [formula (3)] [the term $\Omega^2 x{\cdot}x$ in the Hamiltonian may contain, in particular, an electric potential of the form $\varphi(x) = \frac{e\eta(t)}{2M(t)c^2} x \cdot x$ [6], [18]] for $\Omega \equiv 0$, this reduces to a Landau problem with time-dependent mass M, B. In the Coulomb gauge, the Hamiltonian can be rewritten as the sum of the Hamiltonian of a harmonic oscillator with suitable time-dependent mass μ and frequency ω and of the angular momentum L multiplied by a suitable time-dependent coefficient ν [formula (5)]. In either case the model can be immediately embedded in a three-dimensional one where B is directed along the third direction, and the motion in the direction of the latter is free; the latter would describe, in particular, a charged particle in a suitable t-dependent axially symmetric electromagnetic field (as in [2], but with a possibly t-dependent mass).

The results can be applied also to a two dimensional isotropic harmonic oscillator with time-dependent M, Ω, B on the (canonical) *non-commutative* plane [formula (8)]. In fact, the Hamiltonian can be again rewritten in the same form (5) with new parameters μ, ω, ν involving also the deformation parameter θ of the non-commutative plane in their definition. Non-commutative space-time structure is an old subject dating back over fifty years to Snyder [8]. The analysis of field theories on non-commutative space-times has become a promising area in theoretical physics [9], and also non-commutative quantum mechanics has become subject for intense investigations. In the recent works [10] the formulation, interpretation and applications of non-commutative quantum mechanics have been investigated in configuration space, with the explicit example of the (time-independent) two-dimensional harmonic oscillator in [10], or general Hamiltonians (including the one of the Landau problem) in [11], where an approach to second quantization is also proposed.

In section 2, we define the model and use the socalled Bopp-shift to connect the noncommutative variables to the commutative ones. In section 3, we determine two commuting invariant observables L, I (L is the angular momentum) in terms of some solution of an auxiliary nonlinear ordinary differential equation. In section 4, we determine an orthonormal basis of the Hilbert space consisting of joint eigenvectors of L, I; the presence of the parameter of noncommutativity θ does not modify much the results. Then, we determine suitable time-dependent phase factors such that their products with these joint eigenvectors make up an orthonormal basis of the Hilbert space consisting of solutions of the Schrödinger equation. Each phase factor can be split as the product of a dynamical one and geometric one (including the Berry phase) [**?**]. The latter includes, in general, the effects of the socalled adiabatic operator; the latter multiplied by the socalled dynamical operator and the non-adiabatic one gives a decomposition of the evolution operator, as explained e.g., in the introduction of [7] (see also [15]).

In the rest of this introduction, we briefly recall the general Lewis-Riesenfeld approach to time-dependent quantum mechanics [2]. An operator $I(t)$ is said to be an invariant if

$$\dot{I}(t) \equiv \frac{\partial}{\partial t} I(t) + \frac{1}{i\hbar}\left[I(t), H(t)\right] = 0 \tag{1}$$

(note that H itself is not an invariant unless it is time-independent). Here and in the sequel we use the Schrödinger (*not* the Heisenberg) picture. If $I(t)$ is an invariant and $\psi(t)$ solves the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t}(t) = H(t)\psi(t), \tag{2}$$

then also $I(t)\psi(t)$ does. If in addition $I(t)$ is hermitian, then there exists a (in general time-dependent) orthonormal basis $\{\varphi_{\lambda,\kappa}\}$ of eigenvectors of I with real eigenvalues $\lambda \in \Lambda$; these eigenvectors are parameterized by $\lambda \in \Lambda$ and if necessary by some additional label $\kappa \in K$. Deriving the relation $I\varphi_{\lambda,\kappa} = \lambda\varphi_{\lambda,\kappa}$ with respect to t and using (1), (2) one finds that all eigenvalues λ are time-independent. Moreover, adjusting the phase, the $\varphi_{\lambda,\kappa}$ can be chosen so as to be a complete set of solutions of (2), provided I does not contain the operator ∂_t (for instance, operators of the form $I = \alpha\partial_t$ are not allowed). If H is time-independent and we choose a time-independent I, then eq. (1) amounts to $[I, H] = 0$, and the above scheme amounts to finding a complete set of solutions of (2) in the form of eigenvectors of both I and H, as usual.

2 The model

The Hamiltonian for the two dimensional isotropic harmonic oscillator in a magnetic field reads

$$\begin{aligned} H_{hm}(t) &= \frac{1}{2M(t)}\,[p-eA(t)]\cdot[p-eA(t)]+\frac{M(t)\Omega^2(t)}{2}\,x\cdot x \qquad (3) \\ &= \frac{1}{2M(t)}\left[\left(p_1+\frac{e}{2}B(t)x_2\right)^2+\left(p_2-\frac{e}{2}B(t)x_1\right)^2\right]+\frac{M(t)\Omega^2(t)}{2}\left(x_1^2+x_2^2\right); \end{aligned}$$

we assume piecewise continuously time-dependent mass M, frequency Ω and but magnetic field B (assumed space-independent). The second line holds after expressing the vector potential in Coulomb gauge. As said, for $\Omega\equiv 0$ this reduces to a Landau problem with time-dependent mass M, B,

$$H_L(t)=\frac{[p-eA(t)]\cdot[p-eA(t)]}{2M(t)}=\frac{1}{2M(t)}\left[\left(p_1+\frac{e}{2}B(t)x_2\right)^2+\left(p_2-\frac{e}{2}B(t)x_1\right)^2\right]. \quad (4)$$

We use units such that $\hbar=1$ and x_a, p_a, ω are dimensionless (this can be achieved renormalizing as usual the latter variables by dimensionful factors built out of some constant mass m_0 and frequency ω_0, and $\hbar$). Using the canonical commutation relations $[x_a,x_b]=0=[p_a,p_b]$, $[x_a,p_b]=i\delta_{ab}$ it is straightforwad to check that $H_{hm}(t)$ reduces to

$$\begin{aligned} H(t) &= \frac{1}{2\mu(t)}\,p\cdot p+\frac{\omega^2(t)}{2}\,x\cdot x-\frac{\nu(t)}{2}L \\ &= \frac{1}{2\mu(t)}\left(p_1^2+p_2^2\right)+\frac{\omega^2(t)}{2}\left(x_1^2+x_2^2\right)-\nu(t)L, \qquad (5) \end{aligned}$$

where $L=x_1p_2-x_2p_1$ is the angular momentum, and

$$\mu(t):=M(t),\qquad \omega^2(t):=M(t)\Omega^2(t)+\frac{[eB(t)]^2}{4M(t)},\qquad \nu(t):=\frac{eB(t)}{2M(t)}. \quad (6)$$

Our two-dimensional non-commutative space is defined by noncommuting coordinates $\hat{X}_1,\hat{X}_2$ and associated momentum components $\hat{P}_1,\hat{P}_2$ satisfying the following commutation relations:

$$\left[\hat{X}_1,\hat{X}_2\right]=i\theta,\quad \left[\hat{X}_1,\hat{P}_1\right]=i,\quad \left[\hat{X}_2,\hat{P}_2\right]=i,\quad \left[\hat{P}_1,\hat{P}_2\right]=0. \quad (7)$$

All these operators are hermitian. On such a non-commutative space we consider as a Hamiltonian that of a two dimensional isotropic harmonic oscillator in a magnetic field with t-dependent M,Ω,B,

$$\hat{H}_{hm}(t)=\frac{1}{2M(t)}\left[\left(\hat{P}_1+\frac{e}{2}B(t)\hat{X}_2\right)^2+\left(\hat{P}_2-\frac{e}{2}B(t)\hat{X}_1\right)^2\right]+\frac{M(t)\Omega^2(t)}{2}\left(\hat{X}_1^2+\hat{X}_2^2\right). \quad (8)$$

The aim is to find two independent invariants and then solve the Schrödinger equation within each eigenspace of the latter. We redefine the operators as follows

$$x_1 = \hat{X}_1 + \frac{\theta}{2}\hat{P}_2, \qquad x_2 = \hat{X}_2 - \frac{\theta}{2}\hat{P}_1, \qquad p_1 = \hat{P}_1, \qquad p_2 = \hat{P}_2; \tag{9}$$

the associated commutations relations are

$$[x_1, x_2] = 0, \qquad [x_1, p_1] = i, \qquad [x_2, p_2] = i, \qquad [p_1, p_2] = 0. \tag{10}$$

Such a redefinition is sometimes also called the Bopp shift. In terms of these new operators $\hat{H}_{hm}(t)$ reduces again to (5), where now

$$\mu^{-1} := \frac{[4+eB\theta]^2}{16M} + M\Omega^2\frac{\theta^2}{4}, \qquad \omega^2 := M\Omega^2 + \frac{[eB]^2}{4M}, \qquad -\nu := \frac{[4+eB\theta]eB}{8M} + \frac{M\Omega^2\theta}{2}. \tag{11}$$

(we have omitted the time dependence for brevity). Note that in general μ, ω are t-independent only if all of M, B, Ω are.

Therefore for $B(t) \equiv 0$, we obtain the harmonic oscillator with time-dependent frequency and mass, while for $\Omega(t) \equiv 0$ we obtain the Landau problem with time-dependent frequency and mass, both on the commutative and noncommutative plane.

An operator invariant of the model (5) is an operator that satisfies eq. (1).

3 Derivation of the operator invariants

As $x{\cdot}x$, $p{\cdot}p$, $x{\cdot}p$, $p{\cdot}x$, and therefore H, are explicitly symmetric under rotations, they commute with the generator of rotations L,

$$[L, x{\cdot}x] = 0, \quad [L, p{\cdot}p] = 0, \quad [L, x{\cdot}p] = 0, \quad [L, p{\cdot}x] = 0, \quad [L, H] = 0. \tag{12}$$

Since L is also time-independent, it fulfills (1) and is a first operator invariant. In order to find another one $I(t)$ commuting with L, we make the Ansatz

$$\begin{aligned} I(t) &= \alpha(t)\, x\cdot x + \beta(t)\, p\cdot p + \gamma(t)\,(x\cdot p + p\cdot x) + \delta(t)L \\ &= \alpha(t)\left(x_1^2 + x_2^2\right) + \beta(t)\left(p_1^2 + p_2^2\right) + \gamma(t)\left(\{x_1, p_1\} + \{x_2, p_2\}\right) + \delta(t)L, \end{aligned} \tag{13}$$

where $\{x_1, p_1\} = x_1p_1 + p_1x_1$, $\{x_2, p_2\} = x_2p_2 + p_2x_2$. As we need $I(t)$ to be hermitean, $I^\dagger = I$, the coefficients α, β, γ, δ should be real. As said,

$$[I, L] = 0, \tag{14}$$

because I is manifestly symmetric under rotations. Using (12) and

$$\left[x_1^2, p_1^2\right] = 2i\{x_1, p_1\}, \qquad \left[x_1^2, \{x_1, p_1\}\right] = 4ix_1^2, \qquad \left[p_1^2, \{x_1, p_1\}\right] = -4ip_1^2,$$
$$\left[x_2^2, p_2^2\right] = 2i\{x_2, p_2\}, \qquad \left[x_2^2, \{x_2, p_2\}\right] = 4ix_2^2, \qquad \left[p_2^2, \{x_2, p_2\}\right] = -4ip_2^2,$$

Eq. (1) reduces to

$$\dot{\alpha} - 2\omega^2\gamma = 0, \tag{15}$$
$$\dot{\beta} + \frac{2}{\mu}\gamma = 0, \tag{16}$$
$$\dot{\gamma} + \frac{1}{\mu}\alpha - \omega^2\beta = 0, \tag{17}$$
$$\dot{\delta} = 0. \tag{18}$$

Eq. (18) implies that δ is a constant. The solution of the system (15)-(18) with $\delta = 1$ and all the other coefficients vanishing is the already determined invariant L. In the rest of the text, we choose $\delta = 0$ for the other invariant. From (15)-(17) it follows

$$\frac{d}{dt}\left(\gamma^2 - \alpha\beta\right) = 0, \tag{19}$$

whence

$$\gamma^2 - \alpha\beta = -\kappa^2, \tag{20}$$

where κ is either a real or an imaginary constant (so that κ^2 is real).

Since $I(t)$ and L commute, they have joint eigenvectors. One can find all of the latter using the same operator technique as in Ref. [1], that is the Dirac method of diagonalizing the Hamiltonian of a constant-frequency harmonic oscillator. To this end, we choose a positive κ, stick to positive-definite solutions $\beta(t)$ and introduce an auxiliary real-valued function $\sigma(t)$ such that

$$\beta(t) = \sigma^2(t); \tag{21}$$

then by Eqs. (16) and (20)

$$\gamma = -\mu\sigma\dot{\sigma}, \qquad \alpha = \dot{\sigma}^2\mu^2 + \frac{\kappa^2}{\sigma^2}. \tag{22}$$

Replacing (21), (22) in (17), we find that the system (15-17) is equivalent to the auxiliary equation

$$\ddot{\sigma} + \frac{\dot{\mu}}{\mu}\dot{\sigma} + \frac{\omega^2}{\mu}\sigma = \frac{\kappa^2}{\mu^2\sigma^3}. \tag{23}$$

In terms of a real solution $\sigma(t)$ of (23) the operator invariant (13) takes the form

$$I(t) = (\dot{\sigma}\mu x_1 - \sigma p_1)^2 + \frac{\kappa^2}{\sigma^2}x_1^2 + (\dot{\sigma}\mu x_2 - \sigma p_2)^2 + \frac{\kappa^2}{\sigma^2}x_2^2. \tag{24}$$

4 Eigenstates of $I(t)$ and the phases

We can introduce "raising" and "lowering " operators $a_1,\ a_2,\ a_1^\dagger,\ a_2^\dagger$ by

$$a_a(t) = \frac{1}{\sqrt{2\kappa}}\left(\dot{\sigma}\mu x_a - \sigma p_a + i\frac{\kappa}{\sigma}x_a\right), \qquad a_a^\dagger(t) = \frac{1}{\sqrt{2\kappa}}\left(\dot{\sigma}\mu x_a - \sigma p_a - i\frac{\kappa}{\sigma}x_a\right), \tag{25}$$

with $a = 1, 2$. They fulfill the commutation relations

$$[a_1,\ a_1^\dagger] = 1, \quad [a_2,\ a_2^\dagger] = 1, \quad [a_1,\ a_2] = 0, \quad [a_1,\ a_2^\dagger] = 0. \tag{26}$$

Then the operator invariants I, L can be written as

$$I_-(t) = 2\kappa\left(a_1^\dagger a_1 + a_2^\dagger a_2 + 1\right), \qquad L = i(a_2^\dagger a_1 - a_1^\dagger a_2). \tag{27}$$

Instead of working with the $a_a, a_a^\dagger$ it is convenient to work with the left and right circular annihilation operators A_-, A_+, that are respectively defined by

$$A_-(t) = \frac{1}{\sqrt{2}}\left(a_1 - ia_2\right), \qquad A_+(t) = \frac{1}{\sqrt{2}}\left(a_1 + ia_2\right), \tag{28}$$

and the left and right creation operators, that are the hermitean conjugates of the above:

$$A_-^\dagger(t) = \frac{1}{\sqrt{2}}\left(a_1^\dagger + ia_2^\dagger\right), \qquad A_+^\dagger(t) = \frac{1}{\sqrt{2}}\left(a_1^\dagger - ia_2^\dagger\right). \tag{29}$$

The only non-zero commutators among $A_-,\ A_+,\ A_+^\dagger,\ A_-^\dagger$ are then given by

$$[A_-, A_-^\dagger] = 1, \qquad [A_+, A_+^\dagger] = 1. \tag{30}$$

Using the inverse relations

$$\begin{aligned} a_1 &= \frac{\sqrt{2}}{2}(A_- + A_+), \qquad & a_1^\dagger &= \frac{\sqrt{2}}{2}(A_+^\dagger + A_-^\dagger), \\ a_2 &= \frac{i\sqrt{2}}{2}(A_- - A_+), \qquad & a_2^\dagger &= \frac{i\sqrt{2}}{2}(A_+^\dagger - A_-^\dagger), \end{aligned}$$

we find

$$\begin{aligned} [I_-, A_\pm] &= -2\kappa, \qquad & [I_-, A_\pm^\dagger] &= 2\kappa, \\ [L, A_\pm] &= \pm A_\pm, \qquad & [L, A_\pm^\dagger] &= \mp A_\pm^\dagger, \end{aligned} \tag{31}$$

and

$$I_-(t) = 2\kappa(A_+^\dagger A_+ + A_-^\dagger A_- + 1), \qquad L = (A_-^\dagger A_- - A_+^\dagger A_+). \tag{32}$$

Let us assume $\|0,0\rangle$ is a normalized state annihilated by a_1, a_2, or equivalently by $A_\pm$:

$$A_-\|0,0\rangle = 0, \qquad A_+\|0,0\rangle = 0. \tag{33}$$

For any $r_+,\ r_- \in \mathbb{N}_0$ ($\mathbb{N}_0$ stands for the set of nonnegative integers) we find

$$\begin{aligned} I_-(t)(A_-^\dagger)^{r_+}(A_+^\dagger)^{r_-}\|0,0\rangle &= 2\kappa(r_+ + r_- + 1)(A_-^\dagger)^{r_+}(A_+^\dagger)^{r_-}\|0,0\rangle, \\ L(t)(A_-^\dagger)^{r_+}(A_+^\dagger)^{r_-}\|0,0\rangle &= (r_+ - r_-)(A_-^\dagger)^{r_+}(A_+^\dagger)^{r_-}\|0,0\rangle. \end{aligned} \tag{34}$$

The pairs (r_+, r_-) are in one-to-one correspondence with the pairs (n, m), with n, m defined by

$$n = r_+ + r_- \in \mathbb{N}_0, \qquad m = r_+ - r_- \in \{-n, 2-n, ..., n-2, n\}. \tag{35}$$

The inverse of (35) are

$$r_+ = \frac{n+m}{2} \in \mathbb{N}_0, \qquad r_- = \frac{n-m}{2} \in \mathbb{N}_0. \tag{36}$$

An orthonormal basis of the Hilbert space of states of the system consists of

$$\|n,m\rangle = \mathcal{N}(A_-^\dagger)^{\frac{1}{2}(n+m)}(A_+^\dagger)^{\frac{1}{2}(n-m)}\|0,0\rangle, \qquad \mathcal{N} = \frac{1}{\sqrt{\frac{n+m}{2}!}}\frac{1}{\sqrt{\frac{n-m}{2}!}} \tag{37}$$

which by (34) are eigenvectors of $I_-(t)$ and L:

$$I\|n,m\rangle = 2\kappa(n+1)\|n,m\rangle, \qquad L\|n,m\rangle = m\|n,m\rangle. \tag{38}$$

The action of $A_\pm, A_\pm^\dagger$ on the basis reads:

$$A_+\|n,m\rangle = \sqrt{\frac{n+m}{2}}\,\|n-1,m-1\rangle, \qquad A_+^\dagger\|n,m\rangle = \sqrt{\frac{n+m}{2}+1}\,\|n+1,m+1\rangle, \tag{39}$$

$$A_-\|n,m\rangle = \sqrt{\frac{n-m}{2}}\,\|n-1,m+1\rangle, \qquad A_-^\dagger\|n,m\rangle = \sqrt{\frac{n-m}{2}+1}\,\|n+1,m-1\rangle. \tag{40}$$

By the general theory [2], one can transform the basis $\{\|n,m\rangle\}$ into an orthonormal basis $\{\psi_{n,m}\}$ consisting of solutions of the equation (2) applying suitable phase transformations

$$\psi_{n,m} = e^{i\alpha_{m,n}(t)}\|n,m\rangle \tag{41}$$

By the orthonormality of the basis, the Schrödinger equation

$$i\frac{\partial}{\partial t}\left(e^{i\alpha_{m,n}}\|n,m\rangle\right) = He^{i\alpha_{n,m}}\|n,m\rangle$$

reduces [2] to the following equations for the time-dependent coefficients $\alpha_{m,n}$

$$\frac{d\alpha_{m,n}}{dt} = \langle n, m\| \left(i\frac{\partial}{\partial t} - H \right) \|n, m\rangle. \tag{42}$$

One can thus split $\alpha_{m,n}$ into the sum $\alpha_{m,n} = \alpha^g_{m,n} + \alpha^d_{m,n}$ of a geometric phase (including the Berry phase phenomenon for slowly varying parameters) [12, 13, 14] (see also [16]) and a dynamical phase $\alpha^d_{m,n}$ respectively fulfilling

$$\dot{\alpha}^g_{m,n} = i\langle n, m\| \left(\frac{\partial}{\partial t} \|n, m\rangle \right), \qquad\qquad \dot{\alpha}^d_{m,n} = -\langle n, m\|H\|n, m\rangle. \tag{43}$$

To compute the matrix elements $\langle n, m\|H\|n, m\rangle$, we express H in terms of $A_\pm, A^\dagger_\pm$. As a first step,

$$\begin{array}{ll} x_1 = \frac{-i}{\sqrt{2\kappa}}\sigma(a_1 - a^\dagger_1), & p_1 = \frac{-i}{\sqrt{2\kappa}}\dot{\sigma}\mu(a_1 - a^\dagger_1) - \frac{\sqrt{2\kappa}}{2\sigma}(a_1 + a^\dagger_1), \\ x_2 = \frac{-i}{\sqrt{2\kappa}}\sigma(a_2 - a^\dagger_2), & p_2 = \frac{-i}{\sqrt{2\kappa}}\dot{\sigma}\mu(a_2 - a^\dagger_2) - \frac{\sqrt{2\kappa}}{2\sigma}(a_2 + a^\dagger_2); \end{array} \tag{44}$$

in terms of $A_-, A^\dagger_+, A_+, A^\dagger_-$ we find

$$\begin{aligned} x_1 &= \tfrac{-i\sigma}{2\sqrt{\kappa}} \left(A_- - A^\dagger_+ + A_+ - A^\dagger_- \right), \\ p_1 &= \tfrac{-i\dot{\sigma}\mu}{2\sqrt{\kappa}} \left(A_- - A^\dagger_+ + A_+ - A^\dagger_- \right) - \tfrac{\sqrt{\kappa}}{2\sigma} \left(A_- + A^\dagger_+ + A_+ + A^\dagger_- \right), \\ x_2 &= \tfrac{\sigma}{2\sqrt{\kappa}} \left(A_- - A^\dagger_+ - A_+ + A^\dagger_- \right), \\ p_2 &= \tfrac{\dot{\sigma}\mu}{2\sqrt{\kappa}} \left(A_- - A^\dagger_+ - A_+ + A^\dagger_- \right) - i\tfrac{\sqrt{\kappa}}{2\sigma} \left(A_- + A^\dagger_+ - A_+ - A^\dagger_- \right). \end{aligned} \tag{45}$$

and

$$\begin{array}{ll} x_1 + ix_2 = \frac{i\sigma}{\sqrt{\kappa}} \left(A^\dagger_- - A_+ \right), & p_1 - ip_2 = \frac{i\dot{\sigma}\mu}{\sqrt{\kappa}} \left(A^\dagger_+ - A_- \right) - \frac{\sqrt{\kappa}}{\sigma} \left(A_- + A^\dagger_+ \right), \\ x_1 - ix_2 = \frac{i\sigma}{\sqrt{\kappa}} \left(A^\dagger_+ - A_- \right), & p_1 + ip_2 = \frac{i\dot{\sigma}\mu}{\sqrt{\kappa}} \left(A^\dagger_- - A_+ \right) - \frac{\sqrt{\kappa}}{\sigma} \left(A_+ + A^\dagger_- \right). \end{array} \tag{46}$$

Replacing these formulae in (5) we find

$$\begin{aligned} H &= \frac{1}{2\kappa}\left(\dot{\sigma}^2\mu + \frac{\kappa^2}{\mu\sigma^2} + \omega^2\sigma^2 \right)\left(A^\dagger_- A_- + A^\dagger_+ A_+ + 1 \right) - \nu\left(A^\dagger_- A_- - A^\dagger_+ A_+ \right) \\ &+ \left[\left(\frac{i\sigma\mu}{\sqrt{2\kappa}} + \frac{\sqrt{\kappa}}{\sqrt{2}\sigma} \right)^2 - \frac{\omega^2\sigma^2}{2\kappa} \right] A_- A_+ + \left[\left(\frac{-i\sigma\mu}{\sqrt{2\kappa}} + \frac{\sqrt{\kappa}}{\sqrt{2}\sigma} \right)^2 - \frac{\omega^2\sigma^2}{2\kappa} \right] A^\dagger_+ A^\dagger_-. \end{aligned} \tag{47}$$

Only the operators in the first line contribute to the matrix elements $\langle n,m\|H\|n,m\rangle$, the ones in the second line lead to vanishing terms. So we finally find

$$\langle n,m\|H\|n,m\rangle = \frac{n+1}{2\kappa}\left(\dot\sigma^2\mu + \frac{\kappa^2}{\mu\sigma^2} + \omega^2\sigma^2\right) - m\nu. \tag{48}$$

Now we wish to evaluate

$$\langle n,m\|\frac{\partial}{\partial t}\|n,m\rangle = \mathcal{N}\langle n,m\|\frac{\partial}{\partial t}\left[(A_-^\dagger)^{\frac{1}{2}(n+m)}(A_+^\dagger)^{\frac{1}{2}(n-m)}\|0,0\rangle\right]. \tag{49}$$

$\|0,0\rangle$ is determined up to a time-independent phase.

On one hand, deriving (38) with $n{=}m{=}0$ with respect to t, we find

$$\left(\frac{\partial I}{\partial t}\right)\|0,0\rangle = (2\kappa - I)\left(\frac{\partial}{\partial t}\|0,0\rangle\right), \qquad\qquad L\left(\frac{\partial}{\partial t}\|0,0\rangle\right) = 0. \tag{50}$$

As the left hand-side of $(50)_1$ is different from zero, $\frac{\partial}{\partial t}\|0,0\rangle$ cannot vanish, but must be annihilated by L. We determine it by realizing $a_a, \|0,0\rangle$ in configuration space:

$$a_a = \frac{1}{\sqrt{2\kappa}}\left[\left(\dot\sigma\mu + i\frac{\kappa}{\sigma}\right)x_a + i\sigma\partial_a\right], \qquad\qquad \psi_0(x,t) = \frac{\kappa}{\sigma\sqrt{\pi}}\exp\left[-\left(\frac{\kappa}{\sigma^2} - \frac{i\dot\sigma\mu}{\sigma}\right)\frac{x\cdot x}{2}\right]; \tag{51}$$

in fact ψ_0 is annihilated by both a_1, a_2, therefore also by A_-, A_+, and is normalized w.r.t. the scalar product $(\psi,\psi') = \int d^2x\,\overline{\psi(x)}\psi'(x)$. Deriving $(51)_2$ we find

$$\frac{\partial\psi_0}{\partial t}(x) = \left[\left(\frac{i\ddot\sigma\mu}{\sigma} + \frac{i\dot\mu\dot\sigma}{\sigma} - \frac{i\dot\sigma^2\mu}{\sigma^2} + \frac{2\kappa\dot\sigma}{\sigma^3}\right)\frac{x\cdot x}{2} - \frac{\dot\sigma}{\sigma}\right]\psi_0(x)$$

By the Stokes theorem and some straightforward computations we obtain

$$\int d^2x\,\partial_a(|\psi_0(x)|^2x^a) = 0 \qquad \Rightarrow \qquad \int d^2x\,|\psi_0(x)|^2 x\cdot x = \frac{\sigma^2}{\kappa}\int d^2x\,|\psi_0(x)|^2 = \frac{\sigma^2}{\kappa}.$$

From the two previous equations we conclude

$$\begin{aligned}\langle 0,0\|\left(\frac{\partial}{\partial t}\|0,0\rangle\right) &= \int d^2x\,\overline{\psi_0}\,\frac{\partial\psi_0}{\partial t} = \int d^2x\,\overline{\psi_0}\,\psi_0\left[\left(\frac{i\ddot\sigma\mu}{\sigma} + \frac{i\dot\mu\dot\sigma}{\sigma} - \frac{i\dot\sigma^2\mu}{\sigma^2} + \frac{2\kappa\dot\sigma}{\sigma^3}\right)\frac{x\cdot x}{2} - \frac{\dot\sigma}{\sigma}\right] \\ &= \frac{i\mu}{2\kappa}\left(\sigma\ddot\sigma + \frac{\dot\mu\sigma\dot\sigma}{\mu} - \dot\sigma^2\right). \end{aligned} \tag{52}$$

On the other hand,

$$\begin{aligned}\frac{\partial}{\partial t}\left[(A_-^\dagger)^{\frac{1}{2}(n+m)}(A_+^\dagger)^{\frac{1}{2}(n-m)}\right] &= \frac{n+m}{2}\frac{\partial A_-^\dagger}{\partial t}(A_-^\dagger)^{\frac{1}{2}(n+m-3)}(A_+^\dagger)^{\frac{1}{2}(n-m)} \\ &+ \frac{n-m}{2}(A_-^\dagger)^{\frac{1}{2}(n+m)}\frac{\partial A_+^\dagger}{\partial t}(A_+^\dagger)^{\frac{1}{2}(n-m-2)}; \end{aligned} \tag{53}$$

but from (29) it follows

$$\frac{\partial A_-^\dagger}{\partial t} = \frac{1}{\sqrt{2}}\frac{\partial a_1^\dagger}{\partial t} + \frac{i}{\sqrt{2}}\frac{\partial a_2^\dagger}{\partial t}, \qquad \frac{\partial A_+^\dagger}{\partial t} = \frac{1}{\sqrt{2}}\frac{\partial a_1^\dagger}{\partial t} - \frac{i}{\sqrt{2}}\frac{\partial a_2^\dagger}{\partial t},$$

and by equations (25),

$$\begin{aligned}
\frac{\partial A_-^\dagger}{\partial t} &= \frac{1}{2\sqrt{\kappa}}\left[\left(\ddot{\sigma}\mu + \dot{\mu}\dot{\sigma} + i\frac{\kappa\dot{\sigma}}{\sigma^2}\right)(x_1 + ix_2) - \dot{\sigma}(p_1 + ip_2)\right] \\
&= \frac{i\mu}{2\kappa}\left[\left(\sigma\ddot{\sigma} + \frac{\dot{\mu}\dot{\sigma}}{\mu} - \dot{\sigma}^2\right)A_-^\dagger - \left(\sigma\ddot{\sigma} + \frac{\dot{\mu}\dot{\sigma}}{\mu} - \dot{\sigma}^2 + \frac{2i\kappa\dot{\sigma}}{\mu\sigma}\right)A_+\right], \qquad (54) \\
\frac{\partial A_+^\dagger}{\partial t} &= \frac{1}{2\sqrt{\kappa}}\left[\left(\ddot{\sigma}\mu + \dot{\mu}\dot{\sigma} + i\frac{\kappa\dot{\sigma}}{\sigma^2}\right)(x_1 - ix_2) - \dot{\sigma}(p_1 - ip_2)\right] \\
&= \frac{i\mu}{2\kappa}\left[\left(\sigma\ddot{\sigma} + \frac{\dot{\mu}\dot{\sigma}}{\mu} - \dot{\sigma}^2\right)A_+^\dagger - \left(\sigma\ddot{\sigma} + \frac{\dot{\mu}\dot{\sigma}}{\mu} - \dot{\sigma}^2 + \frac{2i\kappa\dot{\sigma}}{\mu\sigma}\right)A_-\right]. \qquad (55)
\end{aligned}$$

Inserting (52)-(55) into (49) and using (23) we finally find

$$\langle n,m\|\frac{\partial}{\partial t}\|n,m\rangle = i\frac{n+1}{2\kappa}\left(\sigma\ddot{\sigma}\mu + \dot{\mu}\sigma\dot{\sigma} - \dot{\sigma}^2\mu\right) = i\frac{n+1}{2\kappa}\left(\frac{\kappa^2}{\mu\sigma^2} - \omega^2\sigma^2 - \dot{\sigma}^2\mu\right), \qquad (56)$$

as the operators A_+, A_- give vanishing contributions to the scalar product. Replacing (48) and (56) in (42), the latter takes the form

$$\frac{d\alpha_{n,m}}{dt} = -(n+1)\frac{\kappa}{\mu\sigma^2} + m\nu \qquad (57)$$

(there is a partial cancellation of the geometric and dynamical phase derivatives), which is solved by

$$\alpha_{n,m}(t) = -(n+1)\int^t dt'\,\frac{\kappa}{\mu(t')\sigma^2(t')} + m\int^t dt'\,\nu(t'). \qquad (58)$$

5 Conclusions

We have derived a class of exact invariants for the time-dependent, isotropic two dimensional harmonic oscillator in a magnetic field, both on commutative and noncommutative plane. This applies also to the time-dependent Landau problem (which is obtained choosing zero harmonic oscillator frequency $\Omega(t)$). The angular momentum L (absent in the one dimensional case treated in Refs. [1]) is trivially another invariant. Using these two invariants, we have constructed and an orthonormal basis of the associated Hilbert space consisting of solutions of the time-dependent Schrödinger

equation. The presence of the noncommutative parameter θ or of a time-dependent mass does not change too much the results with respect to the commutative space with constant mass, but induces a first order derivative term in the auxiliary equation (23), that is a general form of the Ermakov-Pinney equation in Refs.[17]. When we set $\theta = 0$, then the equation (58) are the same as the results in [2], but in two dimensions. For some choices of $\Omega(t)$, exact solutions of (23) can be found and then the corresponding solutions of the Schrödinger equation in close form. It would be also good to calculate the transition amplitude connecting any initial state in the remote past to any final state in the remote future with constant ω; we hope to report on this point elsewhere.

Note added in proof. In the final stage of publication of the present paper Prof. Dodonov has brought to our attention the very interesting papers,[19], [20], where among other things exact invariants in the form of generalized creation and annihilation operators for a general N-dimensional harmonic oscillator were constructed.

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